



Forman-Ricci Curvature of Tournaments*

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Abstract: tournaments are a type of directed graph which have been used to study the geometry of classical flag manifolds. We became interested in this type of graphs because the combinatorial properties of tournaments can be used to study geometric properties of the flag manifolds. Leal et al [21], Introduced the Forman-Ricci curvature for directed and undirected hypergraphs and obtained the curvature for graphs as a particular case. In this work we present the basic ideas about the Forman-Ricci curvature for directed graphs, characterize the parabolic tournaments in terms of Forman-Ricci curvature and calculate the Forman-Ricci curvature for any tournament.

Keywords: tournaments; Forman-Ricci curvature; parabolic tournaments

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Curvatura de Forman-Ricci de Torneos

Resumen: los torneos son un tipo de grafo dirigido que se han usado para estudiar la geometría de las variedades bandera clásicas. Nos interesamos en este tipo de grafos porque las propiedades combinatorias de los torneos se pueden utilizar para estudiar las propiedades geométricas de las variedades bandera. Leal et al. (2018), Introdujeron la curvatura de Forman-Ricci para hipergrafos dirigidos y no dirigidos y obtuvieron la curvatura para grafos como un caso particular. En este trabajo presentamos las ideas básicas acerca de la curvatura de Forman-Ricci para grafos dirigidos, caracterizamos los torneos parabólicos en términos de la curvatura de Forman-Ricci y calculamos la curvatura de Forman-Ricci para cualquier torneo.

Palabras clave: torneos; curvatura de Forman-Ricci; torneos parabólicos

Curvatura de Forman-Ricci de Torneios

Resumo: os torneios são um tipo de grafo dirigido que tem sido usados para estudar a geometria das variedades bandeira clássicas. Ficamos interessados neste tipo de grafos porque as propriedades combinatórias dos torneios podem ser usadas para estudar as propriedades geométricas das variedades bandeira. Leal et al. (2018), Introduziram a curvatura de Forman-Ricci para hipergrafos dirigidos e não dirigidos obtendo a curvatura para grafos como um caso particular. Neste trabalho apresentamos as ideias básicas sob a curvatura de Forman-Ricci para grafos dirigidos, caracterizamos os torneios parabólicos em termos da curvatura de Forman-Ricci e calculamos a curvatura de Forman-Ricci de qualquer torneio.

Palavras-chave: torneios; curvatura de Forman-Ricci; torneios parabólicos

Introduction

Tournaments are directed graphs which have been used for different geometrical applications; in a tournament (see Moon [1]), each pair of vertices is connected for exactly one and only one oriented arc. Burstall and Salamon [2] were the first mathematicians who used tournaments to study the geometry of flag manifolds; they discovered the relationship between quasi-complex structures on classic flag manifolds and tournaments. With this idea they classified all quasi-complex structures admitting Kähler metrics over classical flag manifolds.

Later, in 2000, Mo and Negreiros [3] applied tournament properties to study (1,2)-symplectic structures on classical flag manifolds and to produce new examples of harmonic maps. From this work arose several works applying similar ideas, among them Cohen, Negreiros and San Martin [4], [5]; Cohen, Paredes and Pinzón [6]; Paredes [7], [8], [9]; Paredes, Gonzalez and McKay [10]; Paredes and Pinzón [11], [12], [13] and [14].

Additionally, during the last decade there has been much research activity on different types of curvatures for graphs or hypergraphs or networks which have been used in biology, finance, new science, among others. For example, in 2011, Lin, Lu and Yau [15] modified the definition of Ricci curvature of Markov chains given by Olivier [16] and introduced the concept of Ricci curvature for graphs. They studied properties of the Ricci curvature of graphs in general, of the Cartesian product of graphs, and of stochastic graphs. In [17], Lin, Lu and Yau studied Ricci-flat graphs, where Ricci curvature vanishes in all its arcs. In [18], Lin and Yau gave some estimates for the Ricci curvature and obtained an estimate for the eigenvalue of the Laplace operator on finite graphs.

Recently, Yamada [19] studied the Ricci curvature of a directed graph, analyzed some properties of the Ricci curvature and found some conditions for a regular directed graph to be flat. Furthermore, he calculated the Ricci curvature of the Cartesian product of directed graphs. Following Yamada's paper, Paredes [20] discussed the Ricci curvature for tournaments, analyzing the case of

parabolic tournaments which Paredes and Pinzón [13] used to study geometric structures on the classic flag manifolds.

Around the same time, Leal, Restrepo, Stadler and Jost [21] introduced a definition of the Forman-Ricci curvature for directed and undirected hypergraphs where the curvature for graphs is a particular case. This definition of curvature was inspired by Riemannian geometry and emphasized the relational character of vertices in a network through the analysis of arcs rather than vertices. The concept of curvature may seem somewhat strange in this context, but there are many recent works that demonstrate the benefits of using this concept, with several interesting applications. Among these works are [21], [22], [23], [24] and [25].

In differential geometry it has been found that curvature accounts for local and global characteristics of differentiable manifolds endowed with a Riemannian metric. Such characteristics generally do not depend on an underlying differentiable structure, and this has led to abstract theories of generalized curvatures in metric spaces. In the case of graphs, these generalized curvatures are relatively easy to define and evaluate and can shed light on other quantities that have been introduced into network analysis without such a clear conceptual background as these curvatures (see [22]). The simplest of these generalized curvatures is the Ricci curvature introduced by Forman for simplicial complexes in [26]. In this work we present the definition and some properties of the Forman-Ricci curvature for directed graphs, calculate this curvature for parabolic tournaments and obtain a formula for the curvature of any n -tournament.

Preliminaries

A graph is a pair $G = (V, E)$, where V is a set of vertices or nodes and E is a set of edges which join pair of vertices. A graph is said to be simple if each edge connects two different vertices and no two edges connect the same pair of vertices, that is, it does not have multiple edges between its vertices. Fig. 1 shows a simple graph.

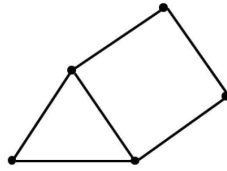


Fig. 1. Simple graph

If the edges also have an orientation, we say that the graph is a directed graph or digraph (see Figure 2), in this case edges are called directed edges or arcs. An arc between two vertices u_1 and u_2 will be denoted as the pair (u_1, u_2) and clearly in this case arcs are ordered pairs. In this article we mainly work with tournaments which are a special type of directed graph.

Definition 1. A tournament or n -tournament is a digraph with n vertices or players without loops such that for each pair of vertices $x \neq y$ there is a unique oriented edge $x \rightarrow y$ or $y \rightarrow x$. If $x \rightarrow y$, then we say x wins against y , see Moon [1].

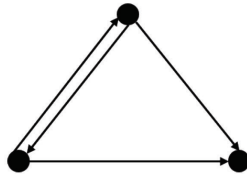


Fig. 2. Directed graph

As pointed out above, we can also represent a tournament as a pair $\tau = (V, E)$, where V is the set of vertices and E is the set of arcs.

Definition 2. Let τ_1 and τ_2 be tournaments with vertices $\{1, \dots, n\}$ and $\{1, \dots, m\}$, respectively. A homomorphism between τ_1 and τ_2 is a mapping

$$\phi: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$$

such that

$$s \xrightarrow{\tau_1} t \implies \left(\phi(s) \xrightarrow{\tau_2} \phi(t) \quad \text{or} \quad \phi(s) = \phi(t) \right)$$

When ϕ is bijective we say that τ_1 and τ_2 are isomorphic, see [1].

Each tournament determines an n -tuple $(s_1, \dots, s_n)$, whose components represent the number of games won by each player. This n -tuple is called score vector; we can order its components in such a way that $0 \leq s_1 \leq \dots \leq s_n \leq n$ and it is easy to see that $\sum_{i=1}^n s_i = \binom{n}{2}$.

Tournaments can be classified by isomorphisms; Figure 3 shows the isomorphism classes of n -tournaments for $n = 2, 3, 4$, together with their score vectors.

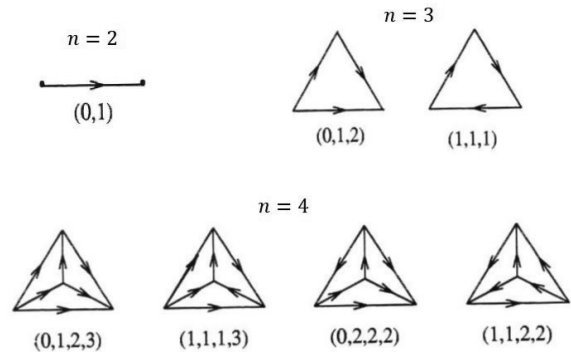


Fig. 3. isomorphism classes of n -tournaments for $n = 2, 3, 4$, and corresponding score vectors

Clearly isomorphic tournaments have the same score vector, but it is not true that tournaments with the same score vector are isomorphic; for example the three 5-tournaments in Figure 4 have the same score vector $(1, 2, 2, 2, 3)$ but they are not isomorphic.



Fig. 4. 5-tournaments with the same score vector

A tournament is transitive if for any three vertices x, y, z such that $x \rightarrow y$ and $y \rightarrow z$ we have $x \rightarrow z$.

All transitive n -tournaments are isomorphic and their score vector is $(0, 1, 2, \dots, n-1)$.

Definition 3. A tournament τ is parabolic if given two vertices i and j , with $j < i$, we have (see [7])

$$\begin{cases} i \xrightarrow{\tau} j, & \text{if } j - i \text{ is odd} \\ j \xrightarrow{\tau} i, & \text{if } j - i \text{ is even} \end{cases} \quad (1)$$

These tournaments received this name because they are related with the parabolic almost complex structures on flag manifolds, see [13]. A complete proof of the following result can be found in [13].

Theorem 1. If is a parabolic n -tournament then its score vector has the form

$$(n - k, \dots, n - k), \text{ if } n = 2k - 1 \quad (2)$$

$$(n - (k + 1), \dots, n - (k + 1), n - k, \dots, n - k), \text{ if } n = 2k \quad (3)$$

Figure 5 contains the parabolic tournaments for 3, 4, 5 and 6 vertices, respectively.

In case $n = 2k$, the converse of the implication is false because, for example, when $n = 6$ there exist non-parabolic tournaments with the same score vector.

When $n = 2k - 1$, our theorem is an equivalence because in this case there is only one isomorphism class whose score vector has all the same components $(n - k, \dots, n - k)$. This is because the sum of the components in the score vector must be

$$1 + 2 + \dots + n - 1 = \frac{(n - 1)n}{2} =$$

$$\frac{(2k - 2)n}{2} = (k - 1)n = (n - k)n$$

Therefore, we have obtained the following result:

Proposition 1. For $n = 2k - 1$, $k = 2, 3, 4, \dots$, $\mathcal{T}$ is a parabolic n -tournament if and only if its score vector has the form $(n - k, \dots, n - k)$.

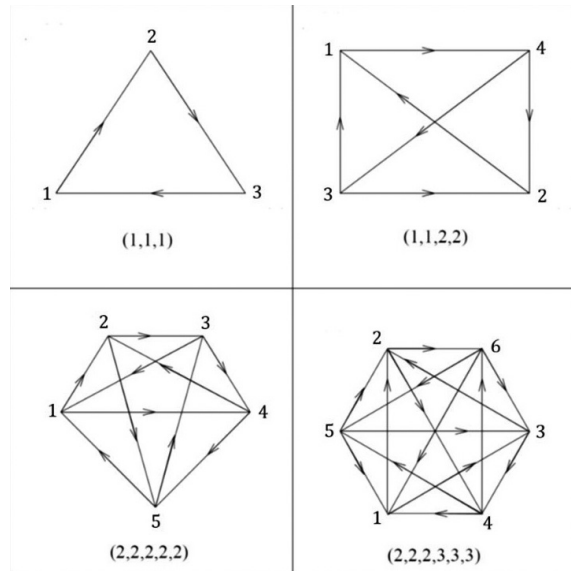


Fig. 5. Parabolic n -tournaments for $n = 3, 4, 5, 6$

Definition 4. For each vertex of a digraph $\mathcal{T} = (V, E)$ we define the out-neighbor of x as the subtournament (see [20])

$$\Gamma^{out}(x) = \{y \in V : (x, y) \in E\} \quad (4)$$

Similarly, we define the in-neighbor of x as the subtournament (see [20])

$$\Gamma^{in}(x) = \{z \in V : (z, x) \in E\} \quad (5)$$

The out-degree of x , denoted by d_x^{out} , is the number of outgoing arcs from x , that is, $d_x^{out} = |\Gamma^{out}(x)|$. We say that $\mathcal{T}$ is d -regular if each vertex has the same out-degree. Because of Proposition 1, parabolic n -tournaments are d -regulars for n odd (see [20]).

Similarly, the in-degree of x , denoted by d_x^{in} , is the number of incoming arcs on x , that is, $d_x^{in} = |\Gamma^{in}(x)|$.

Other type of tournaments which have been used to study the geometry of flag manifolds are locally transitive tournaments. A tournament $\mathcal{T} = (V, E)$ is called locally transitive if the subtournaments $\Gamma^{out}(x)$ and $\Gamma^{in}(x)$ are transitive, see [6], [27], and [28].

Forman-Ricci curvature for directed graphs

Following Leal, Restrepo, Stadler & Jost [21], in this section we present the main ideas of Forman-Ricci curvature for directed graphs and discuss Forman-Ricci curvature of tournaments.

Definition 5. Given a directed graph we consider an arc $e = (u_i, u_j) \in E$, where $u_i, u_j \in V$. For this arc we define (see [21])

$$\text{in-flow at } u_i: F(\rightarrow e) = 1 - d_{u_i}^{\text{in}} \quad (6)$$

$$\text{out-flow at } u_j: F(e \rightarrow) = 1 - d_{u_j}^{\text{out}} \quad (7)$$

The curvature for the flow through the arc $e = (u_i, u_j)$ is given by

$$F(\rightarrow e \rightarrow) = F(\rightarrow e) + F(e \rightarrow) = 2 - d_{u_i}^{\text{in}} - d_{u_j}^{\text{out}} \quad (8)$$

Definition 6. If we consider the loss flow along the arc $e = (u_i, u_j)$ two additional curvatures that account for the flow loss may be calculated (see [21])

$$\text{Flow loss at } u_i: F(\leftarrow e) = 1 - d_{u_i}^{\text{out}} \quad (9)$$

$$\text{Flow loss at } u_j: F(e \leftarrow) = 1 - d_{u_j}^{\text{in}} \quad (10)$$

Therefore, the curvature for the flow-loss along the arc $e = (u_i, u_j)$ is given by

$$F(\leftarrow e \leftarrow) = F(\leftarrow e) + F(e \leftarrow) = 2 - d_{u_i}^{\text{out}} - d_{u_j}^{\text{in}} \quad (12)$$

Finally, the curvature for the total flow over $e = (u_i, u_j)$ is defined as

$$F(e) = F(\rightarrow e \rightarrow) + F(\leftarrow e \leftarrow) \quad (13)$$

Example 1. Consider Figure 6

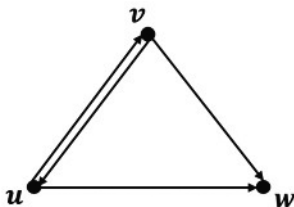


Fig. 6. Graph for Example 1

We have

$$d_u^{\text{in}} = 1, \quad d_u^{\text{out}} = 2, \quad d_v^{\text{in}} = 1 \quad \text{and} \quad d_v^{\text{out}} = 2$$

Hence, for the arc $e = (u, v)$ we obtain

$$F(\rightarrow e) = 1 - d_u^{\text{in}} = 0 \quad \text{and} \quad F(e \rightarrow) = 1 - d_v^{\text{out}} = -1,$$

$$\text{then } F(\rightarrow e \rightarrow) = F(\rightarrow e) + F(e \rightarrow) = -1.$$

Furthermore,

$$F(\leftarrow e) = 1 - d_u^{\text{out}} = -1 \quad \text{and} \quad F(e \leftarrow) = 1 - d_v^{\text{in}} = 0,$$

then $F(\leftarrow e \leftarrow) = F(\leftarrow e) + F(e \leftarrow) = -1$. In conclusion the curvature for the total flow over the arc $e = (u, v)$ is

$$F(e) = F(\rightarrow e \rightarrow) + F(\leftarrow e \leftarrow) = -2$$

Forman-Ricci curvature of tournaments

Now let us see what happens with the Forman-Ricci curvature in the case of tournaments. First, we must observe that isomorphic tournaments have equal curvature, which means that the curvature for each arc will be the same.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two isomorphic tournaments with vertices $\{v_1, v_2, \dots, v_n\}$ and $\{w_1, w_2, \dots, w_n\}$, respectively, and ϕ the corresponding isomorphism. Let's consider $e = (v_i, v_j)$ an arc in $\mathcal{T}_1$ then because of ϕ is an isomorphism we have that $\phi(e) = (\phi(v_i), \phi(v_j)) = (w_i, w_j)$ is an arc in $\mathcal{T}_2$, where $\phi(v_i) = w_i$ and $\phi(v_j) = w_j$.

Moreover, $d_{v_i}^{\text{in}} = d_{w_i}^{\text{in}}$ and $d_{v_i}^{\text{out}} = d_{w_i}^{\text{out}}$, for all $i = 1, 2, \dots, n$, because for example if $d_{v_i}^{\text{in}} \neq d_{w_i}^{\text{in}}$ and $d_{v_i}^{\text{in}} > d_{w_i}^{\text{in}}$ then there must exist a vertex v_k such that $v_k \rightarrow v_i$ but $\phi(v_k) \nrightarrow \phi(v_i)$, which is not possible because ϕ is an isomorphism. Similarly, we can do this for the out-degree of each vertex. Thus, we have proved the following:

Proposition 2. If $\mathcal{T}_1, \mathcal{T}_2$ are isomorphic tournaments, $e = (v_i, v_j)$ is an arc in $\mathcal{T}_1$ and $\phi(e) = (\phi(v_i), \phi(v_j))$ is the arc image in $\mathcal{T}_2$, then $F(e) = F(\phi(e))$.

If we calculate the curvature for 2-tournaments, that is, the tournament in the following figure,

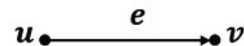


Fig. 7. 2-tournament

we see that

$$F(\rightarrow e) = 1 - d_u^{in} = 1 \quad \text{and} \quad F(e \rightarrow) = 1 - d_v^{out} = 1$$

then $F(\rightarrow e \rightarrow) = 2$.

Similarly,

$$F(\leftarrow e) = 1 - d_u^{out} = 0 \quad \text{and} \quad F(e \leftarrow) = 1 - d_v^{in} = 0$$

then $F(\leftarrow e \leftarrow) = 0$.

Hence, $F(e) = 2$.

Now, we calculate the curvature for the 3-tournament in Figure 8

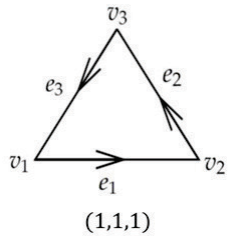


Fig. 8. Parabolic 3-tournament

In this case the calculation is very simple because $d_{v_i}^{in} = 1$ and $d_{v_i}^{out} = 1$, for $i = 1, 2, 3$. Therefore,

$$F(\rightarrow e_i \rightarrow) = 0 \quad \text{and} \quad F(\leftarrow e_i \leftarrow) = 0 \quad \text{then} \quad F(e_i) = 0,$$

for $i = 1, 2, 3$.

Let us consider the other 3-tournament corresponding to the transitive class, see Figure 9. For arc $e_1 = (v_1, v_2)$ we have $d_{v_1}^{in} = 0, d_{v_2}^{in} = 1, d_{v_1}^{out} = 2$ and $d_{v_2}^{out} = 1$, then

$$F(\rightarrow e_1) = 1 - d_{v_1}^{in} = 1 \quad \text{and}$$

$$F(e_1 \rightarrow) = 1 - d_{v_2}^{out} = 0 \quad \Rightarrow \quad F(\rightarrow e_1 \rightarrow) = 1.$$

$$F(\leftarrow e_1) = 1 - d_{v_1}^{out} = -1 \quad \text{and}$$

$$F(e_1 \leftarrow) = 1 - d_{v_2}^{in} = 0 \quad \Rightarrow \quad F(\leftarrow e_1 \leftarrow) = -1$$

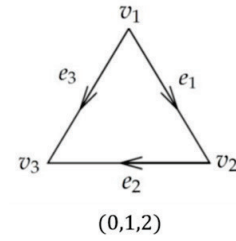


Fig. 9. Transitive 3-tournament

Hence, $F(e_1) = 0$. For arc $e_2 = (v_2, v_3)$ we have $d_{v_2}^{in} = 1, d_{v_3}^{in} = 2, d_{v_2}^{out} = 1$ and $d_{v_3}^{out} = 0$, then

$$F(\rightarrow e_2) = 1 - d_{v_2}^{in} = 0 \quad \text{and} \quad F(e_2 \rightarrow) = 1 - d_{v_3}^{out} = 1$$

$$\Rightarrow \quad F(\rightarrow e_2 \rightarrow) = 1.$$

$$F(\leftarrow e_2) = 1 - d_{v_2}^{out} = 0 \quad \text{and} \quad F(e_2 \leftarrow) = 1 - d_{v_3}^{in} = -1$$

$$\Rightarrow \quad F(\leftarrow e_2 \leftarrow) = -1.$$

Therefore, $F(e_2) = 0$. Finally, for arc $e_3 = (v_1, v_3)$ we have $d_{v_1}^{in} = 0, d_{v_3}^{in} = 2, d_{v_1}^{out} = 2$ and $d_{v_3}^{out} = 0$, then

$$F(\rightarrow e_3) = 1 - d_{v_1}^{in} = 1 \quad \text{and} \quad F(e_3 \rightarrow) = 1 - d_{v_3}^{out} = 1$$

$$\Rightarrow \quad F(\rightarrow e_3 \rightarrow) = 2.$$

$$F(\leftarrow e_3) = 1 - d_{v_1}^{out} = -1 \quad \text{and} \quad F(e_3 \leftarrow) = 1 - d_{v_3}^{in} = -1$$

$$\Rightarrow \quad F(\leftarrow e_3 \leftarrow) = -2.$$

As with the other arcs, for e_3 we also have $F(e_3) = 0$. In other words, all the arcs of a 3-tournament have Forman-Ricci curvature zero; a graph with this property is a flat graph. We will see later that 3-tournaments are the only flat tournaments.

Forman-Ricci curvature of parabolic tournaments

In the last section we see that the Forman-Ricci curvature of the parabolic 3-tournament (Figure 9) is 0. Let us see what happens with the parabolic 4-tournament included in Figure 10. It is easy to see that $d_{v_1}^{in} = 1, d_{v_1}^{out} = 2, d_{v_2}^{in} = 2, d_{v_2}^{out} = 1, d_{v_3}^{in} = 1, d_{v_3}^{out} = 2, d_{v_4}^{in} = 2, d_{v_4}^{out} = 1$. We now calculate the curvature for each arc.

$$\begin{aligned}
 &\text{Arc } e_1 = (v_1, v_2): \\
 &F(\rightarrow e_1) = 1 - d_{v_1}^{in} = 0 \quad F(e_1 \rightarrow) = 1 - d_{v_2}^{out} = 0 \\
 &\Rightarrow F(\rightarrow e_1 \rightarrow) = 0 \\
 &F(\leftarrow e_1) = 1 - d_{v_1}^{out} = -1 \quad F(e_1 \leftarrow) = 1 - d_{v_2}^{in} = -1 \\
 &\Rightarrow F(\leftarrow e_1 \leftarrow) = -2 \\
 &\Rightarrow F(e_1) = F(\leftarrow e_1 \leftarrow) + F(\rightarrow e_1 \rightarrow) = -2.
 \end{aligned}$$

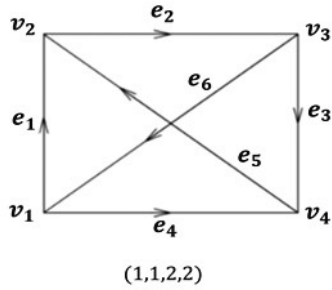


Fig. 10. Parabolic 4-tournament

$$\begin{aligned}
 &\text{Arc } e_2 = (v_2, v_3): \\
 &F(\rightarrow e_2) = 1 - d_{v_2}^{in} = -1 \text{ and } F(e_2 \rightarrow) = 1 - d_{v_3}^{out} = -1 \\
 &\Rightarrow F(\rightarrow e_2 \rightarrow) = -2 \\
 &F(\leftarrow e_2) = 1 - d_{v_2}^{out} = 0 \text{ and } F(e_2 \leftarrow) = 1 - d_{v_3}^{in} = 0 \\
 &\Rightarrow F(\leftarrow e_2 \leftarrow) = 0 \\
 &\Rightarrow F(e_2) = F(\leftarrow e_2 \leftarrow) + F(\rightarrow e_2 \rightarrow) = -2.
 \end{aligned}$$

$$\begin{aligned}
 &\text{Arc } e_3 = (v_3, v_4): \\
 &F(\rightarrow e_3) = 1 - d_{v_3}^{in} = 0 \text{ and } F(e_3 \rightarrow) = 1 - d_{v_4}^{out} = 0 \\
 &\Rightarrow F(\rightarrow e_3 \rightarrow) = 0
 \end{aligned}$$

$$\begin{aligned}
 &F(\leftarrow e_3) = 1 - d_{v_3}^{out} = -1 \text{ and } \\
 &F(e_3 \leftarrow) = 1 - d_{v_4}^{in} = -1 \Rightarrow F(\leftarrow e_3 \leftarrow) = -2 \\
 &\Rightarrow F(e_3) = F(\leftarrow e_3 \leftarrow) + F(\rightarrow e_3 \rightarrow) = -2.
 \end{aligned}$$

Similarly, we can see that $F(e_4) = F(e_5) = F(e_6) = -2$. Then we can say that the Forman-Ricci curvature of the 4-parabolic tournament is -2 .

Remember that Theorem 1 tell us that for a parabolic n -tournament, if $n = 2k - 1$ then its score vector is $(n - k, \dots, n - k)$ and if $n = 2k$ then its score vector is $(n - (k + 1), \dots, n - (k + 1), n - k, \dots, n - k)$. Note that for $n = 3$, the curvature is 0 =

$8 - 4k$, with $k = 2$; and for $n = 4$, the curvature is $-2 = 6 - 4k$, with $k = 2$. We can prove these formulas for all n -tournament.

Theorem 2. If $\mathcal{T}$ is a parabolic n -tournament then its Forman-Ricci curvature is

- $8 - 4k$, if $n = 2k - 1$, for $k = 2, 3, 4, \dots$
- $6 - 4k$, if $n = 2k$, for $k = 1, 2, 3, \dots$

Proof. Let $v_1, \dots, v_n$ be the vertices of the tournament $\mathcal{T}$.

- If $n=2k-1$ the score vector of $\mathcal{T}$ is $(n - k, \dots, n - k)$ then

$$\begin{aligned}
 &d_{v_i}^{in} = n - k, \quad d_{v_i}^{out} = n - k, \\
 &d_{v_j}^{in} = n - k, \quad d_{v_j}^{out} = n - k, \text{ for } k = 2, 3, 4 \dots
 \end{aligned}$$

So, for any arc $e = (v_i, v_j)$ we have

$$F(\rightarrow e) = 1 - d_{v_i}^{in} = 1 - (n - k)$$

and

$$F(e \rightarrow) = 1 - d_{v_j}^{out} = 1 - (n - k).$$

Then,

$$F(\rightarrow e \rightarrow) = F(\rightarrow e) + F(e \rightarrow) = 2 - 2(n - k) = 2 - 2(2k - 1 - k) = 4 - 2k.$$

Similarly, we see that

$$F(\leftarrow e) = 1 - d_{v_i}^{out} = 1 - (n - k) \text{ and}$$

$$F(e \leftarrow) = 1 - d_{v_j}^{in} = 1 - (n - k) \text{ and}$$

$$F(\leftarrow e \leftarrow) = F(\leftarrow e) + F(e \leftarrow) = 4 - 2k.$$

Therefore, $F(e) = 8 - 4k$.

- If $n=2k$ $\mathcal{T}$ is a parabolic n -tournament then its Forman-Ricci curvature is $(n - (k+1), \dots, n - (k+1), n - k, \dots, n - k)$. So, for any arc $e = (v_i, v_j)$ we have two possibilities

$$d_{v_i}^{in} = n - (k + 1), \quad d_{v_i}^{out} = n - k,$$

$$d_{v_j}^{in} = n - k, \quad d_{v_j}^{out} = n - (k + 1),$$

or

$$d_{v_i}^{in} = n - k, \quad d_{v_i}^{out} = n - (k + 1),$$

$$d_{v_j}^{in} = n - (k + 1), \quad d_{v_j}^{out} = n - k,$$

for $k = 1, 2, 3, \dots$ In the first case we get

$$F(\rightarrow e \rightarrow) = 4 - 2k \quad \text{and} \quad F(\leftarrow e \leftarrow) = 2 - 2k \\ \Rightarrow F(e) = 6 - 4k,$$

and in the second case we get

$$F(\rightarrow e \rightarrow) = 2 - 2k \quad \text{and} \quad F(\leftarrow e \leftarrow) = 4 - 2k \\ \Rightarrow F(e) = 6 - 4k.$$

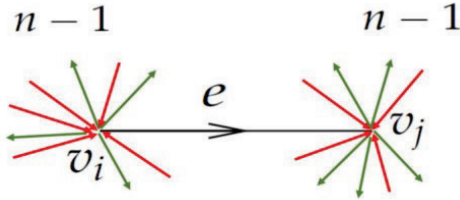


Fig. 11. Arc of an n -tournament

Forman-Ricci curvature of any tournament

In the two previous sections we calculated the Forman-Ricci curvature for 2-tournaments and 3-tournaments; from these calculations we observe that the sum of the in-degrees plus the out-degrees of the vertices of an arc is a constant.

For example, in the case of the 2-tournament in Figure 7 we have

$$d_u^{in} + d_v^{in} + d_u^{out} + d_v^{out} = 0 + 1 + 1 + 0 = 2$$

Similarly, for the parabolic 3-tournament (Figure 8), $d_{v_i}^{in} = 1$ and $d_{v_i}^{out} = 1$, for $i = 1, 2, 3$. Then, for any arc $e = (v_i, v_j)$ we have

$$d_{v_i}^{in} + d_{v_j}^{in} + d_{v_i}^{out} + d_{v_j}^{out} = 1 + 1 + 1 + 1 = 4$$

In the case of the transitive 3-tournament (Figure 9) we have three possibilities

- Arc : $e_1 = (v_1, v_2)$: $d_{v_1}^{in} + d_{v_2}^{in} + d_{v_1}^{out} + d_{v_2}^{out} = 0 + 1 + 2 + 1 = 4$,
- Arc : $e_2 = (v_2, v_3)$: $d_{v_2}^{in} + d_{v_3}^{in} + d_{v_2}^{out} + d_{v_3}^{out} = 1 + 2 + 1 + 0 = 4$,
- Arc : $e_3 = (v_1, v_3)$: $d_{v_3}^{in} + d_{v_1}^{in} + d_{v_3}^{out} + d_{v_1}^{out} = 2 + 0 + 0 + 2 = 4$.

Also, it is easy to see that for 4-tournaments like the one in Figure 10 for each arc $e = (v_i, v_j)$ it is true that

$$d_{v_i}^{in} + d_{v_j}^{in} + d_{v_i}^{out} + d_{v_j}^{out} = 6.$$

Therefore, for each arc $e = (v_i, v_j)$ of any n -tournament,

$$d_{v_i}^{in} + d_{v_i}^{out} = n - 1 \quad \text{and} \quad d_{v_j}^{in} + d_{v_j}^{out} = n - 1,$$

as is shown in Figure 11. This is because given two vertices v_i and v_j in an n -tournament there exists only one arc from v_i to v_j or vice versa. For that reason, the sum of arcs going in and out a vertex is a constant and equal to $n - 1$ (the number of won games plus the number of lost games is equal to $n - 1$).

Theorem 3. If $\mathcal{T}$ is an n -tournament, then its Forman-Ricci curvature for any arc $e = (v_i, v_j)$ is

$$F(e) = 6 - 2n \quad (14)$$

Proof. Directly calculating the curvature of the arc $e = (v_i, v_j)$ we have

$$F(e) = F(\rightarrow e \rightarrow) + F(\leftarrow e \leftarrow)$$

$$F(e) = (2 - d_{v_i}^{in} - d_{v_j}^{out}) + (2 - d_{v_i}^{out} - d_{v_j}^{in})$$

$$F(e) = 4 - (d_{v_i}^{in} + d_{v_i}^{out}) - (d_{v_j}^{in} + d_{v_j}^{out})$$

$$F(e) = 4 - (n - 1) - (n - 1)$$

$$F(e) = 6 - 2n.$$

It is interesting that the Forman-Ricci curvature of tournaments gives very simple values and most of them are negative, as seen in the following table

Table 1. Forman-Ricci curvature of n -tournaments

Number of vertices n	Forman-Ricci curvature
$n = 2$	$F(e) = 2$
$n = 3$	$F(e) = 0$
$n = 4$	$F(e) = -2$
$n = 5$	$F(e) = -4$
$n = 6$	$F(e) = -6$
$\vdots$	$\vdots$
n	$F(e) = 6 - 2n$

As we mentioned before, we see that the only flat tournaments for the Forman-Ricci curvature are the 3-tournaments.

Final remarks

This work suggests new ideas for novel research, for example the possibility of using graph curvature concepts to explore and obtain new geometric properties of flag manifolds.

There are different curvatures for graphs, but we do not know which best relates to the geometry of the flag manifolds. Possibilities include the Ricci curvature for directed graphs introduced by Yamada [19], the Ollivier's Ricci curvature [16] used to study and analyze complex networks, and the different curvature concepts proposed by Jost and collaborators [22], [23], [24] and [29].

Recently, some works on energy of directed graphs have been published, see for example [30], [31] and [32]. These works use the out-degree and the in-degree. We intend to study these works to relate the results obtained here with the concept of energy of digraphs studied in the mentioned works.

There are several applications for those curvature concepts mentioned above. For example, Farooq et al. [33] studied the structural networks of the brain using a geometric approach based on the Ollivier-Ricci curvature. This approach also has been used to assess the robustness of the financial market and to differentiate the biological networks of cancer cells from healthy ones. They applied curvature-based measurements to the brain's structural networks to identify robust and fragile brain regions in healthy subjects, demonstrating that curvature can also be used to track age-related changes in brain connectivity and Autism Spectrum Disorder (ASD). The results obtained are in accordance with previous magnetic resonance imaging (MRI) studies.

Pouryahya, Mathews, and Tannenbaum [34] use three different types of curvature to study biological networks (Ollivier-Ricci curvature, Bakry-Émery Ricci curvature, and Forman-Ricci curvature). While the exact relationship between these three definitions of curvature is not yet

known, the authors report that all three curvatures produced similar results for the studied biological networks. For instance, they studied cancer networks generated by TCGA (The Cancer Genome Atlas) data and classified and compared the major genes in breast carcinoma with respect to the three notions of curvature. Genes are classified based on the difference in curvature between cancer and normal networks (without cancer). In fact, these are the nodes that are expected to have the greatest contribution to the robustness of the cancer network compared to the normal network. The seven cancer networks studied showed higher average curvature than normal complementary networks for all three types of curvature.

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References

- [1] J. W. Moon, Topics on tournaments, Holt, New York: Rinehart and Winston, 1968.
- [2] F.E. Burstall y S. Salamon, "Tournaments, Flags and Harmonic Maps", *Mathematische Annalen*, vol. 277, pp. 249-265, 1987. DOI: <https://doi.org/10.1007/BF01457363>
- [3] X. Mo y C.J.C. Negreiros, "(1,2)-symplectic structures on flag manifolds", *Tohoku Mathematical Journal*, Second Series, vol. 52, n.o 2, pp. 271-282, 2000. DOI: <https://doi.org/10.2748/tmj/1178224611>
- [4] N. Cohen, C. J. C. Negreiros y L. A. B. San Martin, "(1, 2)-Symplectic metrics, flag manifolds and tournaments", *Bulletin of the London Mathematical Society*, vol. 34, pp. 1-9, 2002. DOI: <https://doi.org/10.1112/S002460930200142X>
- [5] N. Cohen, C. J. C. Negreiros y L. A. B. San Martin, "A rank-three condition for invariant (1,2)-symplectic almost Hermitian structures on flag manifolds", *Bulletin of the Brazilian Mathematical Society*, New Series, vol. 33, n.o 1, pp. 49-73, 2002. DOI: <https://doi.org/10.1007/s005740200002>
- [6] N. Cohen, M. Paredes y S. Pinzón, "Locally transitive tournaments and (1,2)-symplectic metrics on maximal flag manifolds", *Illinois Journal of Mathematics*,

- vol. 48, n.o 4, pp. 1405-1415, 2004. DOI: <https://doi.org/10.1215/ijm/1258138518>
- [7] M. Paredes, “Aspectos da Geometria Complexa das Variedades Bandeira”, Tese de Doutorado, Universidade Estadual de Campinas, Brasil, 2000.
- [8] M. Paredes, “Some results on the geometry of full flag manifolds and harmonic maps”, *Revista Colombiana de Matemáticas*, vol. 34, pp. 57-89, 2000.
- [9] M. Paredes, “Families of (1,2)-symplectic metrics on full flag manifolds”, *International Journal of Mathematics and Mathematical Sciences*, vol. 29, pp. 651-664, 2002. DOI: <https://doi.org/10.1155/S0161171202012267>
- [10] M. Paredes, P. González y B. McKay, “Sobre un tipo especial de torneos y una clase de métricas sobre variedades bandera”, in *Memorias de la Primera Conferencia Iberoamericana de Matemática Computacional*. Bogotá: Sociedad Colombiana de Matemáticas, 2001, pp. 125-137.
- [11] M. Paredes y S. Pinzón, “On the codifferential of the Kähler form and cosymplectic metrics on maximal flag manifolds”, *Turkish Journal of Mathematics* vol. 34, 305-316, 2010.
- [12] M. Paredes y S. Pinzón, “Some remarks about cosymplectic metrics on maximal flag manifolds”, in *Geometric and Topological Methods in Quantum Field Theory*, H. Ocampo, E. Pariguán and S. Paycha, eds. Cambridge: Cambridge University Press, 2010, pp. 394-404. DOI: <https://doi.org/10.1017/CBO9780511712135.013>
- [13] M. Paredes y S. Pinzón, “Tournaments and parabolic almost complex structures on flag manifolds”, *Contemporary Mathematics*, vol. 509, pp. 221-231, 2010. DOI: <https://doi.org/10.1090/conm/509/09981>
- [14] M. Paredes y S. Pinzón, “Variedades bandera maximales, torneos y aplicaciones armónicas”, *Revista Integración*, vol 18, n.o 2, pp. 65-77, 2000.
- [15] Y. Lin, L. Lu y S.T. Yau, “Ricci curvature of graphs”, *Tohoku Mathematical Journal, Second Series*, vol. 63, n.o 4, pp. 605-627, 2011. DOI: <https://doi.org/10.2748/tmj/1325886283>
- [16] Y. Ollivier, “Ricci curvature of Markov chains on metric spaces”, *Journal of Functional Analysis*, vol. 256, pp. 810-864, 2009. DOI: <https://doi.org/10.1016/j.jfa.2008.11.001>
- [17] Y. Lin, L. Lu y S.T. Yau, “Ricci-flat graphs with girth at least five”, 2013. [Internet]. Available: <https://arxiv.org/abs/1301.0102>
- [18] Y. Lin y S.T. Yau, “Ricci Curvature and eigenvalue estimate on locally finite graphs”, *Mathematical Research Letters*, vol. 17, pp. 343-356, 2010. DOI: <https://doi.org/10.4310/MRL.2010.v17.n2.a13>
- [19] T. Yamada, “The Ricci curvature on directed graphs”, *Journal of the Korean Mathematical Society*, vol. 56, n.o 1, pp. 113-125, 2019. DOI: <https://doi.org/10.4134/JKMS.j180088>
- [20] M. Paredes, “Torneos parabólicos, torneos localmente transitivos y la curvatura de Ricci para torneos parabólicos”, *Realidad y Reflexión*, vol. 51, 142-157, 2020. DOI: <https://doi.org/10.5377/ryr.v51i0.9858>
- [21] W. Leal, G. Restrepo, P.F. Stadler y J. Jost, Forman-Ricci Curvature for Hypergraphs, 2018. [Internet]. Available: <https://arxiv.org/abs/1811.07825>
- [22] A. Samal, R. P. Sreejith, J. Gu, S. Liu, E. Saucan, and J. Jost, “Comparative analysis of two discretizations of Ricci curvature for complex networks”, *Scientific Reports*, vol. 8, n.o 1, pp. 1-16, 2018. DOI: <https://doi.org/10.1038/s41598-018-27001-3>
- [23] E. Saucan, A. Samal, M. Weber, and J. Jost, “Discrete curvatures and network analysis”, *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 80, no. 3, pp. 605-622, 2018. https://match.pmf.kg.ac.rs/electronic_versions/Match80/n3/match80n3_605-622.pdf
- [24] R. P. Sreejith, J. Jost, E. Saucan, and A. Samal, “Systematic evaluation of a new combinatorial curvature for complex networks”, *Chaos Solitons and Fractals*, vol. 101, pp. 50-67, 2017. DOI: <https://doi.org/10.1016/j.chaos.2017.05.021>
- [25] M. Weber, J. Stelzer, E. Saucan, A. Naitsat, G. Lohmann, and J. Jost, “Curvature based methods for brain network analysis”, 2017. [Internet]. Available: <https://arxiv.org/abs/1707.00180>
- [26] R. Forman, “Bochner’s method for cell complexes and combinatorial Ricci curvature”, *Discrete and Computational Geometry*, vol. 29, n.o 3, pp. 323-374, 2003. DOI: <https://doi.org/10.1007/s00454-002-0743-x>
- [27] A. E. Brouwer, “The enumeration of locally transitive tournaments”, *Afdeling Zuivere Wiskunde [Department of Pure Mathematics]*, Mathematisch Centrum, Amsterdam, vol. 138, 1980.
- [28] J.S. Fuentes, M. Paredes and S. Pinzón, “Acerca de los dígrafos localmente transitivos”, *Revista Integración*, vol. 22, pp. 23-35, 2004.
- [29] R.P. Sreejith, K. Mohanraj, J. Jost, E. Saucan and A. Samal, “Forman curvature for complex networks”, *Journal of Statistical Mechanics: Theory and Experiment*, vol. 6, p. 063206, 2016. DOI: <https://doi.org/10.1088/1742-5468/2016/06/063206>

- [30] N. Agudelo, J. Monsalve and J. Rada, “On the characterization of digraphs with given rank”, *Linear Algebra and its Applications*, vol. 587, pp. 215-227, 2020. <https://doi.org/10.1016/j.laa.2019.11.008>
- [31] J. Monsalve and J. Rada, “Oriented bipartite graphs with minimal trace norm”, *Linear Multilinear Algebra*, vol. 67, no. 6, pp. 1121-1131, 2019. DOI: <https://doi.org/10.1080/03081087.2018.1448051>
- [32] J. Monsalve, J. Rada and Y. Shi, “Extremal values of energy over oriented bicyclic graphs”, *Applied Mathematics and Computation*, vol. 342, pp. 26-34, 2019. DOI: <https://doi.org/10.1016/j.amc.2018.09.018>
- [33] H. Farooq, Y. Chen, T. T. Georgiou, A. Tannenbaum and C. Lenglet, “Network curvature as a hallmark of brain structural connectivity”, *Nature communications*, vol. 10, no. 1, pp. 1-11, 2019. DOI: <https://doi.org/10.1038/s41467-019-12915-x>
- [34] M. Pouryahya, J. Mathews, y A. Tannenbaum, “Comparing three notions of discrete Ricci curvature on biological networks”, 2017. arXiv preprint arXiv:1712.02943.